Teaching with Technology in Undergraduate Mathematics

Resource Book Part 3
Supporting Student Inquiry

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Creating a Culture of Inquiry

A culture of inquiry in the classroom means that students are inquiring about mathematics and instructors are supporting their students' inquiry.

Culture of inquiry allows students to be at the center and take ownership of their own learning.

Deeply Engage Students

A culture of inquiry requires students that are deeply engaged in mathematics. You should move beyond simple calculation problems and give tasks which engage the whole learner. Tasks that are "low-floor high-ceiling" are especially supportive of inquiry learning. Low-floor high-ceiling tasks are tasks that a wide variety of students can access (low-floor), and at the same time the mathematics has the potential to be extremely deep (high-ceiling). Students find these tasks intrinsically motivating. Often, problems related to real-world applications or future career-goals can get students to engage deeply.

Create Opportunities to Collaborate

Collaboration and mutual appreciation is crucial to promoting inquiry. Students need to see each other as productive mathematical problem-solvers. By assigning tasks that require different skills sets and assigning roles to participants in groups, you can create opportunities to collaborate where each student plays a crucial role

Center Student Thinking

It is essential that student thinking is recognized, appreciated, and valued in the classroom. If all mathematical authority rests on the instructor, then students will not see their roles as important to their learning. One technique for centering student thinking is to name theorems and lemmas in class after the student who first discovered them.

Instructor Support for Student Equity

Tang et al. (2017, p.S10) outline the ways that inquiry learning can be connected to equity. This includes:

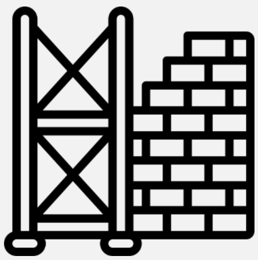
- the instructor developing an understanding of student thinking and supporting student thinking,
- creating a community where all students have access to the mathematics,
- an environment where students create their own knowledge about mathematics,
- an environment where the tasks build on prior knowledge and understanding,
- creating a community where each student has the opportunity to participate and act mathematically including using generalization and justification of ideas, and
- creating an environment where each student has the opportunity to be successful.

Create a Moment of Surprise - The Inquiry Moment

Inquiry requires motivation to understand or explain something. When students are involved in inquiry, they want to know. Thus, each inquiry activity needs to begin with a question or a moment of surprise. One way of doing this in the mathematics classroom is using mathematical paradoxes.

Making Sure to Summarize Learning

Inquiry is all about exploration which is really important. This means though, that in a good inquiry activity many different ideas are explored, and the mathematical ideas can be experienced episodically (even when relationships are developed during the inquiry activity). Therefore, it is really important to consolidate and anchor the learning with a summative discussion. The summative discussion can move from the concrete to the more abstract- starting with a summary of big ideas and moving to applications and relationships.



Scaffolding Student Thinking

No one can understand a big idea all in one go. Scaffolding is taking a larger idea and breaking it down into more manageable parts.

When you scaffold student thinking you support the student until they are able to work on and think about the idea independently.

Here are **5 techniques** to help you use scaffold student learning in your teaching:

Thinking-Aloud/ Modelling Thinking

In thinking aloud the instructor models (narrates) thinking through the topic or problem. This should be prepared for in advance. The thinking modelled does not have to necessarily be the instructor's own thinking. This is because instructors have already developed expertise in the area and their thinking might be too advanced or nuanced for beginners. Instead, the instructor should put themselves in the minds of their students and think about an efficient, correct, and the easiest to follow process.

Pre-Teach or Prepare Students with Definitions/Language

Language can be a barrier to accessing ideas. Preparing students before a lesson with the major language they will be using will help lighten their cognitive load for the mathematical idea. This strategy is especially helpful for second language learners.

Give Time to Process New Information

Ideas in a mathematics class build on one another. Students need time at each level to process the information. As students do this, they will begin to connect between ideas and learn the ideas at a deeper level. Without time to process, learning will remain superficial, and students will have a hard time drawing connections and applying the knowledge.

Take Breaks (Pause and Reflect)

Taking breaks goes right along with giving time to process new information. As an instructor you can place "pause and reflect" moments into your teaching agenda. During these moments you can use a number of different strategies: think/pair/share or four corners for example.

Use Multiple Representations

This is one of the themes of this course mainly because learning an idea through multiple representations is so powerful in scaffolding student thinking. Through multiple representations students interact with one idea through different lenses.

Tips for Starting Class Discussions

Two ingredients make for a good discussion: the choice of an appropriate strategy and a good prompt or question!

We explore a number of different strategies for a classroom discussion, which could be face to face or online.

Adapting Questions

Although “information recall” questions are important, they should not be a primary way of asking questions. When you adapt information recall questions, they will resonate better with your students.

- Instead of asking “What does the Intermediate Value Theorem say,” ... you could ask “What does the fact that the domain of the function is closed tell you?”
- Replace a “Can you define” question with a “compare” question, for instance “What do symmetric and triangular matrices have in common? How are they different?”
- Ask “What if” questions, such as “What happens to this statement if you replace such-as-such assumption by this one?”
- Instead of asking a usual “explain” question, specify the level, such as “How would you explain this to a grade 5 student?” or “How would you explain this to someone who does not know what a function is?”
- Instead of asking a student to recall a formula, ask them to narrate it in words, without using symbols.

Using "Provocative" Questions and Inquiry:

Creative, unexpected, provocative, questions will spark your students' curiosity and increase their motivation to engage (for more discussion see the section on inquiry p.4)

- After a student solves a problem, you can say: "Good, it's correct. But can you create a similar, but harder, problem and solve it? [We routinely ask students to solve problems, but not often enough to formulate problems]
- Forbid (rule out) options. Students solved a problem correctly. Then you present a challenge: "Can you come up with an alternative solution, but now you are not allowed to use algebra?"
- Create a proof and then delete several lines or statements from it, including the concluding line. Ask your students to complete the proof, and to discover the statement that they were proving. [This could lead to multiple correct answers]
- Creatively warn students not to do "mechanical-looking" things mechanically, without thinking. For instance, give them a formula for a function whose domain is empty, and ask them to compute its derivative. [Although the calculation can be carried out, it makes no sense]
- Give students a complex question and ask them to devise a plan to solve it: suggest that they break it down into parts, describe what these parts are, and how they would put them together to solve the problem.
- Give students a topic, say eigenvectors, and ask them to create 3 or 4 good test questions about it.

Think Pair Share

Think Pair Share stimulates students to think individually, share ideas, refine their understanding, and formulate an answer by consensus (or disagreement!). You can use Think Pair Share at the start of your class to activate prior knowledge, any time to check students' understanding, or at the end of class to summarize main points of your lecture. You can use this strategy to ask students to hypothesize - for instance, stop in the middle of a lecture, and ask -"What should we do next?". One wonderful aspect of this strategy is that it could be used when time is short (5-6 minutes). However, if you do use this strategy when time is short, be sure to make sure that the "think" part is there.

Think pair share could be challenging in large classes: noise levels will go up, you will not be able to monitor what many students are doing- so you will not have time to stimulate students who are not engaged to actually do something. Also, it might take some time to "bring the students back" to your lecture after a Think Pair Share.

5 Minute Polls

Polls are an efficient way to feel the "pulse" of your classroom. If your lecture uses concepts defined in your previous lecture(s), a poll can be a good way for you to figure out where your students are - should you dive right into the new material, or spend some time reviewing? You can use poll questions to encourage students to familiarize themselves with the course outline and other course-related information; or, to ask them about their readiness for the next test. You can include fun questions in your poll, to relax your students. For instance, "where would you like to be now?" (with an option "of course, in this math class").

Four Corners

Four Corners was typically used in face-to face teaching to promote discussion. In a face-to-face class, students are asked a questions and have the option of four answers. These four answers are placed on poster board in each corner of the room. For example, students can be given the statement "All areas on an infinite interval will yield an infinite area." Students would then choose one of the corners around the room: either corner 1) Completely Agree, corner 2) Somewhat Agree, corner 3) Somewhat Disagree, or corner 4) Completely Disagree. Once the students are in their corner they discuss why they are there. Someone is elected to keep notes and then share the discussion with the class.

The online method of doing this activity is pretty much the same as face to face, except we use breakout rooms instead of corners. Once a question is asked, students have a choice of 4 breakout rooms. They are told in advance which breakout room represents which "corner."

Waterfall

Waterfall is a really quick strategy that students love. All you do is tell students you are going to ask them a question. They will need to type their answers in the chat box, but should not hit enter until you say "go." Give the students a couple of minutes to type their answers. Then say go. All the students will hit the return button at the same time. The answers will cascade down the chat like a waterfall. As the instructor, you read out and engage with some of the answers.

Strategies for Group Work

Different types of learning occur when students collaborate compared to when they work alone.

You will need to decide based on your students and course/lesson objectives which one is right for your activities.

Appoint Roles (Accountability)

Appointing student roles during group work can help promote accountability and increase student agency. You might appoint roles such as:

- **Class presenter**: this student will report back to the class on the group's final solution to the problem.
- **Solution writer**: this student will prepare a written version of the solution for submission.
- **Liaison**: this student will visit other groups and report back on their progress toward solving the problem.

Reporting Back

Without the necessity to report back to the instructor, or the class, group work can feel isolating. By asking students to report back to the instructor or class, they know that their work will be heard and discussed by other people. Reporting back creates a sense of community and allows students to consolidate their learning, as they compare strategies. When asking students to report back:

- inform students before they start working that they'll need to report back after working,
- allow appropriate time so that most groups have something to report back, and,
- offer meaningful commentary once a group has reported, and you might also want to have students compare and contrast reports.

Display Work

Displaying student work is a good alternative to having students report back. It can help alleviate time pressure and allow groups to interact with each other without needing to report to the whole class. Some strategies for displaying work:

- write work on an interactive whiteboard,
- create an "impromptu poster session" where students display their work around the classroom, and/or
- write up final solutions and ask students to share them in the chat simultaneously.

Full-Class Summary Discussion

It is important that students leave a lesson understanding what the main idea of the lesson was. Full-class summary discussions allow the group work to be consolidated through a discussion around the "so whats" and "now whats" of the lesson. When preparing a lesson that has a full-class summary discussion:

- inform groups they'll need to share to the whole class,
- set aside appropriate time for each group to present, and
- respond to and acknowledge each group's progress.

Strategies for Individual Work

Different types of learning occur when students collaborate compared to when they work alone.

You will need to decide based on your students and course/lesson objectives which one is right for your activities.

Self-Check with Technology

When assigning individual work to a class of students working online, it is possible to ask students to do a self-check using technology. After students have worked on a calculation problem, you can ask them to confirm their final answer using tools such as Desmos or WolframAlpha. If the students' answers agree with the online calculators, it is quite likely that their calculations and reasoning are correct. If there is a mistake, then it is interesting to discuss what might have caused the mistake. You can have the discussion as a group, with yourself and the student, or you can do a think pair share

Full-Class Summary Discussion

After students complete an individual task, it can be helpful to lead a full-class discussion of the task. During such a discussion, be sure to:

- Center student thinking: allow students to explain their thought processes and talk about their approaches to the problem.
- Promote multiple answers: attempt to find different solutions and approaches to the problem

Dealing with Student Anxiety

Anxiety can present itself in many different ways. Students are often embarrassed to share that they are experiencing anxiety. Many students experience (non-visible) anxiety. Studies have shown up to ⅔ of students experience anxiety in university. Through being aware of responses we can be inclusive of all students in our classes.

Accept Student Feelings (Don’t Dismiss)

Students are often embarrassed to share that they are experiencing anxiety. Students make themselves very vulnerable when they do share what they are experiencing. It is very important therefore to accept the student’s sharing, and not dismiss their concerns. The student is experiencing the fear as real, so the first strategy is to show acceptance that the student is feeling anxiety. And of course, always talk to the student in a non-public setting.

Create Community and Encourage Students

Anxiety is often invisible in the classroom. Competitive environments increase anxiety. One way to help anxiety and create community is to eliminate open competition. In class, encourage students and remind them that their entire degree and their worth is not determined by the one class. Also, be available for students and make sure to set up time for one-on-one interactions with your students.

Clear Communication

There are two things to consider with communication. First, being clear and calm with communication and expectations helps to relieve a lot of student anxieties. Plan out communication before you send an email or make a course announcement. Anything ambiguous or unclear will create anxiety on behalf of the students and of course anxiety means more work for you as the emails come pouring in. Second, communicate often. Use group emails to remind students of important milestones and dates in the course, to send updates, and just to “touch base.”

Know to Whom to Refer Students

Every university has student resources for students who are experiencing anxiety. As a course instructor/ TA you can and should only deal with what you can. Become aware of the resources your university offers so you can refer students who need those resources.

Preparing Questions and Resources

Learning is stronger and lasts longer when students work out a solution to a problem themselves. Although students still learn when we as instructors tell them an answer, their learning is not as strong.

An instructor's role is to guide students through the problem-solving process by asking good questions at appropriate times.

The 5 P's of Questions

There are different types of questions you could ask. Student thinking and response will differ depending on the type of question. The 5 P's of question-asking are:

- Pace questions: e.g., What can you tell me about...?
- Probing questions: e.g., Why? Can you tell me more about that?
- Prodding questions: e.g., What do you think it is?
- Prompting questions: e.g., What do you think would happen if…?
- Process questions: e.g., How would you compare this concept with…?

Questions that Elicit Student Voice

Sometimes you want to know how a student arrived at an answer, or you may need more explanation to understand what the student is thinking, or you may need to intervene to help a student understand the mathematics. You can use these questions to understand or push student thinking and ideas:

- What is the question asking?
- What does this mathematical terminology mean? What is the definition of XX?
- Have you seen a similar example elsewhere? Can you modify it to answer the question?
- Is there a way to avoid this lengthy calculation you have started?

- What do you think you should do next?
- What techniques (of calculating XX) do you know?
- What reasoning strategies could you use? Can you use an example, or a couple of examples, to show that this is true?
- What's your plan to solve this problem?

Questions that Advance Students' Thinking

Sometimes you want to scaffold (for more discussion see the section on scaffolding p.6) or push your students thinking further. You can use these questions to help push your students thinking further:

- Does the solution seem reasonable? Does it make sense? If it is an application question: What does your model predict will happen? How realistic is it?
- Can you use rough estimation to check your numeric answer?
- Is there an alternative method to solve this? Have you tried a graphical method?
- Can you explain what you did and why you did it?
- If there are units: are they consistent, did you do the dimensional analysis?
- "What if" questions: the function is defined on a closed interval. What if the interval were not closed?

General Strategies

- Revisit previous mathematical knowledge and ask students to review a previous mathematical topic.
- Help students break down the problem into smaller problems, and ask them how the solutions to these smaller problems will help them solve the initial problem.
- Use and connect mathematical representations: ask students to draw a diagram or use other visual supports to explain and justify their reasoning.
- Direct students to appropriate resources (be as specific as possible, especially when referring to online resources).
- Ask students to validate their answer (explain to them what "validate" means).
- In class, plan for sufficient wait time so students can formulate and offer responses; if online, ask them not to post their replies in chat before everyone had a chance to think about the question.

Question Posing to Explore Theorems

The development of reasoning skills and proof techniques often comes from studying mathematical theorems.

The first step to guide students through such a mathematical journey is to pose good questions when introducing a theorem.

Key Ideas

Mathematical theorems not only form the foundation of mathematics but also carry mathematical knowledge and insights that help students develop a deeper understanding of underlying concepts. Proof is not only the heart of mathematics, but the pathway to creating analytic tools and catalyzing growth. However, it is an elusive concept for many mathematics students, and it is difficult to teach and learn. To foster students' understanding of proofs in online environments is a challenging task. Learning to write proofs often starts with reading them, i.e., reading proofs that others have done. Proof comprehension, although challenging, can be achieved in both face-to-face and online environments.

General Considerations

Posing good questions about theorems and proofs is one of the effective ways to help students appreciate, grasp, and work with theorems appropriately. Keep in mind that theorems and working with theorems are new to all students. They especially struggle with the logical structure of the implication, since logic is not covered in high school. It is important to understand what the theorem says before starting a proof. Thus, students should engage with numerous examples, especially with geometric and visual ones. Students also should be encouraged to create their own examples. Emphasize that a theorem cannot be proved by examining several examples. However, we can disprove a theorem by picking one counterexample.

Type of Questions to Pose When Teaching a New Theorem

When teaching a new theorem, your questions should be framed around:

- Helping your students understand all the terms involved;
- Articulating what is assumed and what is claimed to be true;
- Understanding and applying the logical structure of the theorem;
- Seeing the relevance of the theorem.

Questions that help your students understand all the terms involved

- Read the theorem carefully. What is a continuous function? What does "between" mean, can you express it in terms of math inequalities?

Questions that articulate what is assumed and what is claimed to be true

- What are the assumptions of this theorem, and what is its conclusion? Can you state/write the theorem in your own words, without using math symbols?
- Can you illustrate the assumptions and the conclusion of the theorem in an example? Draw a sketch

Questions that develop understanding and application of the logical structure of the theorem

- Does the conclusion of the theorem hold for discontinuous functions?
- Note that f is defined on a closed interval [a,b]. Does the conclusion of the theorem hold true if the interval is not closed?
- Can you think of an algebraic example to illustrate this theorem?

Questions that help your students to see the relevance of the theorem

- What does the theorem tell us in the case when $f(a)<0$, $f(b)>0$ and lambda=0? What could be a possible use of this?

Scaffolding Questions to Help Students Understand a Proof of a Theorem

Proof is a tremendously challenging concept for students, who struggle to make sense of the purposes for, structures of, or validity criteria of proofs. Teaching strategies that focus heavily on the presentation of formal proofs result in rote and routinized approaches to proving that leave students with little ability to produce anything independently other than trivial proofs. Teaching strategies need to model and emphasize the process of proving so that students can develop the competencies and deductive reasoning necessary for mathematics. Engage students in proof explorations and proving activities such as scientific debate, or constructing proofs and presenting them to peers. Scaffolding questions is a critical strategy that helps mitigate difficulties.

Questions that examine a proof locally:

- What are the assumptions?
- Clearly state what we are supposed to prove.

- Read the proof carefully, sentence by sentence, line by line. How does this line in calculation follow from the previous line?
- Ask pointed questions: Why does the point X lie on the line Y? Provide an appropriate evidence.
- Suggest: Verify that X = Y. What's the relevance of this equality, i.e., how is it used in the proof?
- Relate to diagrams, graphs: This diagram shows that the two graphs intersect at three points. Show that this is indeed true.

Questions that examine a proof globally:

- What's the method used in this proof? Is it a direct proof, proof by contradiction or proof by the contrapositive?
- What's the critical step of this proof? Would it work without the assumptions made in the theorem?
- What's the key idea of this proof? Can you state it in words, without using symbols and formulas?
- Think of other theories we have proved so far. Is this proof similar in structure to any of them?

A Long Path to Understanding

Many students find proofs challenging is due to their limited understanding of the proof structure and the laws of formal (logical) reasoning.

- Inform students that proofs can be difficult to grasp and that no single course suffices to teach proofs. In fact, they will be learning how to prove in most of their undergraduate math courses.
- Invite students to go over the given proof to determine the method used (direct proof, proof by contradiction, or proof by contrapositive).
- Invite students to articulate, in their own words and without symbols, what is assumed and what needs to be proved.
- Invite students to start their own proof by listing all assumptions (now in math terms, using symbols etc.)
- Local understanding: ask students to read a proof, line by line, and examine how the arguments are established. Suggest making notes, adding comments, doing side calculations, making sketches, diagrams, etc. Passively reading someone else's proof, as it's a piece of literature, does not contribute to understanding.
- Global understanding: find an alternative proof, locally understand it, and compare the two proofs.

Student Generated Example Spaces

Students learn mathematics through definitions, proofs, techniques, but even more important is that they learn from engagement with examples. The broader the range of examples, the richer the possibilities are for mathematical generalizations.

Student generated examples (Watson & Mason, 2002) are central to them building understanding of the underlying mathematical ideas in larger concepts.

Why Example Spaces?

Learning mathematics can be thought of as a process of generalizing from specific examples and can be seen as growth and extension of personal, situated, example spaces. Personal example spaces are particularly influential in the construction of one's conceptual understanding.

There are 5 types of examples that are critical to learning mathematics

- examples illustrating concepts,
- examples demonstrating methods and techniques,
- examples of appropriate objects which satisfy certain conditions,
- examples of ways of constructing proof, and
- examples of disproving a conjecture.

Personal Example Spaces

Personal example spaces consist of: examples, counter-examples, and non-examples of mathematical objects or diagrams of mathematical objects. In a mathematical context, the same example may play different roles:

- |x| is an example of a continuous function on the reals, but a counterexample to the conjecture that all continuous functions are also everywhere differentiable
- 0.9 is a non-example of a number whose square is larger than it, or a counterexample to the conjecture that "squaring makes larger."

Experiencing Structure

These types of example spaces ask students to construct and deconstruct mathematical ideas in order to "experience the mathematical structure." In Watson and Mason's article, they had students construct and deconstruct linear equations, so the students could experience the underlying elements of linear equations.

- Strategies to support student learning: Students can work in small groups or individually. Students need to keep track of the various ideas that have been deconstructed so they should use an interactive whiteboard or some other sort of tool that allows the student to "see" the various parts that they have deconstructed and are now using to reconstruct the mathematical idea.

Experiencing and Extending the Range of Variation

These example spaces give the student the opportunity to play with different representations of the same mathematical idea. The representations can be entirely different mediums (as we have shown throughout this course- interactives, symbolic, etc.) and it can be experiencing different equations that lead to the same answer.

Strategies to support student learning: The idea here is to give students lots of different opportunities with different medium. The various modules in our course will help you decide what to use when. In introducing this in class, you can have different groups of students working on different representations. After working, the students would come together in the end to explain their representation to the rest of the class. Or you can systematically have students work with different representations and compare them.

Experiencing Generality

These types of example spaces asks students to notice patterns emerging from the examples they are working with.
Strategies to support student learning: Watson and Mason (2002) recommend 3 steps:

- Particular: In this step students generate multiple examples from a question. This step generates a range of examples.
- Peculiar: Once students have established a pattern they are asked to identify something peculiar. Once students have identified a range of examples, they can then search for the peculiar among them.
- General: The students then use the range of examples and the peculiar examples to make a mathematical generalization.

Experiencing the Constraints and Meanings of Conventions

In these types of example spaces students learn the constraints and why we use conventions in mathematics. Whatever mathematical topic you teach, some aspect will be based on convention that students just have to accept. In this example space students learn why the convention is efficient and the limitations the convention may have.

Strategies to support student learning: The most important strategy for this example space is in the post-activity debriefing. The debriefing should consist of an analysis of the different examples with a discussion of the limitations and affordances of the different methods. The conclusion of the debriefing tackles why that specific convention was chosen to be the universal convention.

Extending Example-Spaces and Exploring Boundaries

These types of example spaces systematically explore the parameters of a mathematical idea. For example, in their article Watson and Mason explore quadrilaterals with students. They first asked students to draw a quadrilateral. Most students drew a rectangle. To explore the parameters of quadrilaterals they gave a sequence of steps: a quadrilateral with a pair of sides equal; a quadrilateral with a pair of sides equal and a pair of sides parallel; a quadrilateral with all these features and a pair of opposite angles equal.

Strategies to support student learning: The strategies for this example space lie in the questioning and planning out the sequence of questions. Questions should move from the general (like draw a quadrilateral) incrementally toward the specific. Students use different examples at different times- it all depends on what triggers the examples. We could help students extend and enrich their example spaces by:

- making use of a sequence of tasks and interactions,
- demonstrating possible variations and ranges of permissible changes,
- exemplifying invariance in the midst of change, and
- inviting student to construct examples subject to constraints.